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On Cauchy–Kovalewski Extension

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Let $\mathscr{A}$ be the universal Clifford algebra of the euclidean space $\mathbb{R}^m$, $e_1, ..., e_m$ the euclidean basis, $D = \partial/\partial x_0 + \sum_{i=1}^m e_i \, \partial/\partial x_i$ the main operator of Clifford analysis. Given f: $\mathbb{R}^m \to \mathbb{R}$ a certain kind of rational function, it is possible to find explicitly the extension of f, f^*: $\mathbb{R}^{m+1} \to \mathscr{A}$, satisfying $Df^* = 0$. This also gives the harmonic extension of f. f^* is given by integral representation on simplex in $\mathbb{C}^n$. © 1991 Academic Press, Inc.

Introduction

In [1], the Cauchy–Kovalewski extension was defined. It is known how to extend polynomials and hence how to extend real analytic functions.

Here, we show how to extend rational functions which have a factorization in the product of functions of first degree.

The motivations of that problem are as follows:

— there are no good methods of harmonic extension of rational functions

— rational functions are very important in the theory of one complex variable. What are their roles in Clifford analysis?

— the Dirac equation with an external electromagnetic field would be better understood if we knew how to manage multiple singularities.

Notation

Recall some notations from [1]. $\mathscr{A}$ is the universal Clifford algebra of the euclidean space $\mathbb{R}^m$ but with the minus sign for the scalar product. $e_1, ..., e_m$ is the euclidean basis and

$$D = \frac{\partial}{\partial x_0} + \sum_{i=l}^{m} e_i \frac{\partial}{\partial x_i}$$

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is the generalized Cauchy–Riemann operator acting on functions defined on the $\mathbb{R}^{m+1}$ space. A function is (left) monogenic when $Df=0$,

Let $\Omega \subset \mathbb{R}^m$ be open and $f\colon \Omega \to \mathbb{R}$.

The maximal (left) monogenic extension $f^*\colon U \to \mathscr{A}$, U open in $\mathbb{R}^{m+1}$, $f^* = f$ on Ω, $Df^* = 0$, is called the (left) Cauchy–Kovalewski extension.

$\mathscr{A}$ is thought to be imbedded in his complexified $\mathscr{A} \otimes_{\mathbb{R}} \mathbb{C}$.

The standard p-simplex is denoted by

$$\Sigma_p = \left\{ t^{(p)} \in \mathbb{R}^p;\ \sum_{j=1}^{p} t_j \leqslant 1 \text{ and } 0 \leqslant t_j \text{ for all } j \right\};$$

elements of $\mathbb{R}^p$ are $t^{(p)} = (t_1, ..., t_p)$,

$$A_{p-1} = \left\{ t^{(p)} \in \mathbb{R}^p;\ \sum_{j=1}^{p} t_j = 1 \text{ and } 0 \leqslant t_j \text{ for all } j \right\}.$$

Let $v_1, ..., v_k \in \mathscr{A}$ and $z = (z_1, ..., z_k) \in \mathbb{C}^k$; then $\langle z, v \rangle = \sum_{j=1}^{k} z_j v_j$,

$$\omega(z) = dz_1 \wedge \cdots \wedge dz_k$$

$$\omega'(z) = \sum_{l=1}^{k} (-1)^{l-1} z_l\, dz_1 \wedge \cdots \wedge \widehat{dz_l} \wedge \cdots \wedge dz_k.$$

Note that $d\omega'(z) = k\omega(z)$.

Let $j = (j_1, ..., j_k) \in \mathbb{N}^k$ a multiindice; then $|j| = j_1 + \cdots + j_k$, $z^j = z_1^{j_1} \cdots z_k^{j_k}$, $j! = j_1! \cdots j_k!$. Let $\alpha = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k$; then $D^{\alpha}_{\zeta,0} = \partial^{\alpha_1 + \cdots + \alpha_k} / \partial\zeta_1^{\alpha_1} \cdots \partial\zeta_k^{\alpha_k}$ is taken for $\zeta = 0$.

If E is a set, VE is a neighborhood of E.

1. Homotopy and Clifford-Valued Integrals

In this section, we define, study, and compute some Clifford valued integrals based on deformations of simplex.

Lemma 1. *Let $a, b, c \in \mathscr{A}$ and $p, q \in \mathbb{N}$ such that $p \geqslant q$ and $v \subset \mathbb{C}$ a closed path such that $a + zb + (1-z)c$ is invertible for all $z \in v$. Then*

$$\int_v \frac{(1-z)^q\, dz}{[a + zb + (1-z)c]^{p+2}} = 0.$$

Proof. The Clifford algebra $\mathscr{A}$ may be thought of as a matrix algebra; then the singularities of $1/[a + zb + (1-z)c]^{p+2}$ are isolated. Then, we must show only that for all singularities z_0 there exists a neighborhood $V\{z_0\}$ of z_0 such that

$$\int_v \frac{(1-z)^q\, dz}{[a+zb+(1-z)c]^{p+2}} = 0$$

with $v \subset V\{z_0\}$.

The first case is for $b-c$ invertible:

$$\begin{aligned}
&\frac{1}{a+zb+(1-z)c}\\
&\qquad = \frac{1}{(z-z_0)(b-c)+z_0(b-c)+a+c}\\
&\qquad = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(z-z_0)^{k+1}} (b-c)^{-1}[z_0+(b-c)^{-1}(a+c)]^k.
\end{aligned}$$

We can take $V\{z_0\}$ such that this series is uniformly convergent on $V\{z_0\}$,

$$\begin{aligned}
&\int_v \frac{(1-z)^q\, dz}{[a+zb+(1-z)c]^{p+2}}\\
&= \int_v \left[\sum_{k=0}^{\infty} (-1)^k \frac{1-z_0-(z-z_0)}{(z-z_0)^{k+1}} (b-c)^{-1}(z_0+(b-c)^{-1}(a+c))^k\right]^q\\
&\quad \times \left[\sum_{h=0}^{\infty} (-1)^h \frac{1}{(z-z_0)^{h+1}} (b-c)^{-1}(z_0+(b-c)^{-1}(a+c))^h\right]^{p-q+2} dz = 0
\end{aligned}$$

because we have no term involving $1/(z-z_0)$. ∎

Proposition 1. *Let $m_1, ..., m_k$, k invertible elements of the Clifford algebra $\mathscr{A}$. There exists C^∞ mapping*

$$\gamma_p\colon\ V\Sigma_p \to \mathbb{C}, \qquad 1 \leqslant p \leqslant k-1,$$

such that

$$\gamma_p(1, 0, ..., 0) = 1;$$

$$\gamma_p(0, t_2, ..., t_p) = 0 \qquad \textit{for all} \quad (t_2, ..., t_p) \in \Sigma_{p-1}$$

$$\begin{aligned}
\Gamma\colon\ & V\Sigma_{k-1} \to \mathbb{C}^{k-1}\\
& t^{(k-1)} \mapsto (\gamma_{k-1}(t^{(k-1)}), ..., \gamma_1(t^{(1)}))
\end{aligned}$$

$z_1 m_1 + (1-z_1)(z_2 m_2 + (1-z_2)(\dots(z_{k-1} m_{k-1} + (1-z_{k-1}) m_k)\dots)$ is invertible for all $z_p \in \gamma_p(t^{(p)})$. The integral

$$I = \int_{\Gamma(\Sigma_{k-1})} \frac{(1-z_1)^{k-2}(1-z_2)^{k-3}\cdots(1-z_{k-2})\, dz_1 \wedge \cdots \wedge dz_{k-1}}{\{z_1 m_1 + (1-z_1)(z_2 m_2 + (1-z_2)(\dots(z_{k-1} m_{k-1} + (1-z_{k-1}) m_k)\dots)\}^k}$$

does not depend on the particular choice of the maps $\gamma_1, \dots, \gamma_{k-1}$.

Proof. Once again we use the isolated singularity property: there exists a path

$$\gamma_1 \colon V\Sigma_1 \to \mathbb{C}, \qquad \gamma_1(0) = 0, \quad \gamma_1(1) = 1,$$

such that $\zeta m_1 + (1-\zeta) m_2$ is invertible on this path. Hence, the integral

$$\int_{\gamma_1(\Sigma_1)} \frac{d\zeta}{[\zeta m_1 + (1-\zeta) m_2]^2} = \int_{\Sigma_1} \frac{\gamma_1'(t)\, dt}{[\gamma_1(t)\, m_1 + (1-\gamma_1(t)) m_2]^2}$$

is well defined, and by Lemma 1 does not depend on the map γ_1.

This proposition is true for $k=1$. Suppose that it is also true for $k-1$: Fix $t^{(k-1)} \in \Sigma_{k-1}$; there exists a path

$$\gamma_k(\cdot, t^{(k-1)}) \colon V[(0, t^{(k-1)}), (1, 0)] \to \mathbb{C}$$

such that

$$\gamma_k(0, t^{(k-1)}) = 0, \qquad \forall t^{(k-1)} \in \Sigma_{k-1}$$
$$\gamma_k(1, 0) = 1$$

and, letting

$$m(t^{(k-1)}) = \gamma_{k-1}(t^{(k-1)}) m_1 + (1 - \gamma_{k-1}(t^{(k-1)}))(\gamma_{k-2}(t^{(k-2)}) m_2 + \cdots + (1 - \gamma_1(t^{(1)}) m_k) \cdots),$$

$\zeta m_0 + (1-\zeta)\, m(t^{(k-1)})$ is invertible on that path.

The following integral is well defined:

$$I = \int_{\gamma_1(\Sigma_1)} \int_{\gamma_2(\Sigma_2)} \cdots$$
$$\cdots \int_{\gamma_{k-1}(\Sigma_{k-1})} \frac{(1-z_0)^{k-1} dz_0 \wedge (1-z_1)^{k-2} \cdots (1-z_{k-2}) dz_1 \wedge \cdots \wedge dz_{k-1}}{[z_0 m_0 + (1-z_0)(\cdots)]^{k+1}}.$$

By induction this integral does not depend on the particular path $\gamma_2, \dots, \gamma_{k-1}$ and by Lemma 1 it does not depend on the particular path γ_1. ∎

PROPOSITION 2. *There exists a $\mathscr{C}^\infty$ map ψ: $V\Sigma_{k-1} \to \mathbb{C}^{k-1}$ such that*

$$\psi(0, ..., 0) = (0, ..., 0); \qquad \psi(0, ..., 0, 1, 0, ..., 0) = (0, ..., 0, 1, 0, ..., 0);$$

$\zeta \in \psi(\Sigma_{k-1}) \Rightarrow \sum_{j=1}^{k-1} \zeta_j m_j + (1 - \sum_{j=1}^{k-1} \zeta_j)\, m_k$ *invertible. The integral of Proposition* 1 *is*

$$I = \int_{\psi(\Sigma_{k-1})} \frac{d\zeta_1 \wedge \cdots \wedge d\zeta_{k-1}}{\left[\sum_{j=1}^{k-1} \zeta_j m_j + \left(1 - \sum_{j=1}^{k-1} \zeta_j\right) m_k\right]^k}.$$

Proof. Change the variables in Proposition 1:

$$\begin{aligned}
\zeta_1 &= z_1 \\
\zeta_2 &= (1 - z_1) z_2 \\
&\vdots \\
\zeta_{k-1} &= (1 - z_1)(1 - z_2) \cdots (1 - z_{k-2}) z_{k-1}.
\end{aligned}$$

Then

$$\begin{aligned}
d\zeta_1 \wedge \cdots \wedge d\zeta_{k-1} = dz_1 \wedge ((1 - z_1)\, dz_2 - z_2\, dz_1) \wedge \cdots \\
\wedge \left(\sum_{j=1}^{k-2} (1 - z_1) \cdots \widehat{(1 - z_j)} \cdots (1 - z_{k-2})\, z_{k-1}(-1)\, dz_j\right. \\
\left. + (1 - z_1) \cdots (1 - z_{k-2})\, dz_{k-1}\right) \\
= (1 - z_1)^{k-2} \cdots (1 - z_{k-2})\, dz_1 \wedge \cdots \wedge dz_{k-1}.
\end{aligned}$$

The coefficient of m_1 is $z_1 = \zeta_1$, m_2 is $(1 - z_1) z_2 = \zeta_2$, and so on, ..., of m_{k-1} is $(1 - z_1)(1 - z_2) \cdots (1 - z_{k-2}) z_{k-1} = \zeta_{k-1}$, of m_k is $(1 - z_1) \cdots (1 - z_{k-2})(1 - z_{k-1}) = 1 - \zeta_1 - \zeta_2 - \cdots - \zeta_{k-1}$.

Composing the map Γ of Proportion 1 and this change of variables, we get a map

$$\begin{aligned}
&V\Sigma_{k-1} \to \mathbb{C}^{k-1} \\
&t^{(k-1)} \mapsto (\gamma_{k-1}(t^{(k-1)}), ..., [1 - \gamma_{k-1}(t^{k-1})] \cdots [1 - \gamma_2(t^{(2)})]\, \gamma_1(t^{(1)}))
\end{aligned}$$

which is a $\mathscr{C}^\infty$ mapping such that

$$(0, ..., 0) \mapsto (0, ..., 0)$$

$$(0, ..., 0, 1, 0, ..., 0) \mapsto (0, ..., 0, 1, 0, ..., 0)$$

and $\sum_{j=1}^{k-1} \zeta_j m_j + (1 - \sum_{j=1}^{k-1} \zeta_j) m_k$ is invertible for ζ belonging to the image of that map. Let ψ be this map. ∎

THEOREM 1. *Let $m_1, \ldots, m_k$, k invertible elements of the Clifford algebra $\mathscr{A}$. There exists a simplex $\Lambda \subset \mathbb{C}^k$, defined by the image of A_{k-1} by a $\mathscr{C}^\infty$ mapping such that*

(i) *$z \in \Lambda \Rightarrow \langle z, m\rangle$ is invertible in $\mathscr{A}$*

(ii) *$I = (-1)^{k-1} \int_\Lambda \omega'(z)/\langle z, m\rangle^k$ does not depend on the particular choice of Λ.*

Proof. Consider two changes of variables:

$$f\colon \mathbb{C}^{k-1} \to \mathbb{C}^k$$
$$\zeta \mapsto z$$
$$z_1 = \zeta_1, \ldots, z_{k-1} = \zeta_{k-1}, \qquad z_k = 1 - \sum_{j=1}^{k-1} \zeta_j$$
$$\tilde{f}\colon \Sigma_{k-1} \to A_{k-l}$$
$$t \mapsto x$$
$$x_1 = t_1, \ldots, x_{k-1} = t_{k-1}, \qquad x_k = 1 - \sum_{j=1}^{k-1} t_j.$$

Let $\tilde{\psi} = \phi \circ \psi \circ \tilde{f}^{-1}$. We have

$$\sum_{p=1}^{k} (-1)^{p-1} z_p \, dz_1 \wedge \cdots \wedge \widehat{dz_p} \wedge \cdots dz_k = (-1)^{k-1} \, d\zeta_1 \wedge \cdots \wedge d\zeta_{k-1}.$$

Hence, the integral I of Proposition 2 is equal to

$$I = (-1)^{k-1} \int_{\tilde{\psi}(A_{k-1})} \frac{\omega'(z)}{\langle z, m\rangle^k}.$$

We can put $\Lambda = \tilde{\psi}(A_{k-1})$.

The integral I does not depend on the particular choice of Λ because it is possible to proceed in the opposite way; starting with a Λ we make the inverse change of variables, getting the integral of proposition 2 and the integral of Proposition 1 which do not depend on the particular path. ∎

PROPOSITION 3. *Let $a_1, \ldots, a_k$, k strictly positive real numbers. Then*

$$(-1)^{k-1}(k-1)! \int_{A_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k} = \frac{1}{a_1 \cdots a_k}.$$

Proof. The differential form $\omega'(x)/\langle x, a\rangle^k$ is closed:

$$\begin{aligned} d\frac{\omega'(x)}{\langle x, a\rangle^k} &= k\frac{dx_1 \wedge \cdots \wedge dx_k}{\langle x, a\rangle^k} - k\frac{1}{\langle x, a\rangle^{k+1}} d\langle x, a\rangle \wedge \omega'(x) \\ &= k\frac{dx_1 \wedge \cdots \wedge dx_k}{\langle x, a\rangle^k} - k\frac{1}{\langle x, a\rangle^{k+1}} \sum_{j=1}^k a_j\, dx_j \wedge \omega'(x) \\ &= k\frac{dx_1 \wedge \cdots \wedge dx_k}{\langle x, a\rangle^k} - k\frac{1}{\langle x, a\rangle^{k+1}} \sum_{j=1}^k a_j x_j \omega(x) \\ &= 0. \end{aligned}$$

Let

$$\begin{aligned} B_{k-1} &= \left\{x \in \mathbb{R}^k;\ \sum_{j=1}^k a_j x_j = 1, 0 \leqslant x_j\right\} \\ \Omega &= \{x \in \mathbb{R}^k; x = t\alpha + (1-t)\beta, t \in [0, 1], \alpha \in A_{k-1}, \beta \in B_{k-1}\} \\ C_j &= \{x \in \mathbb{R}^k; x_j = 0\} \cap \Omega. \end{aligned}$$

The boundary of Ω is $A_{k-1} \cup B_{k-1} \cup \bigcup_j C_j$. On C_j, $x_j = 0$; hence $\omega'(x) = 0$:

$$0 = \int_\Omega d\frac{\omega'(x)}{\langle x, a\rangle^k} = -\int_{A_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k} + \int_{B_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k} - \sum_{j=1}^k \int_{C_j} \frac{\omega'(x)}{\langle x, a\rangle^k}.$$

Hence

$$\int_{A_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k} = \int_{B_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k}.$$

We set

$$y_j = a_j x_j \Leftrightarrow x_j = \frac{y_j}{a_j}$$

$$\int_{B_{k-1}} \frac{\omega'(x)}{\langle x, a\rangle^k} = \int_{A_{k-1}} \frac{\omega'(y/a)}{\sum_{j=1}^k y_j} = \frac{1}{a_1 \cdots a_k} \int_{A_{k-1}} \omega'(y)$$

$$\int_{A_{k-1}} \omega'(y) = (-1)^{k-1} \int_{\Sigma_{k-1}} dy_1 \wedge \cdots \wedge dy_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}. \quad \blacksquare$$

PROPOSITION 4. *Let $a_1, ..., a_k \in \mathbb{C}\backslash\{0\}$ and $\Lambda \subset \mathbb{C}^k$ a simplex image of A_{k-1} by a $\mathscr{C}^\infty$ mapping such that $z \in \Lambda \Rightarrow \langle z, a\rangle \neq 0$. Then*

$$(-1)^{k-1}(k-1)! \int_\Lambda \frac{\omega'(z)}{\langle z, a\rangle^k} = \frac{1}{a_1 \cdots a_k}.$$

Proof. Theorem 1 prove that such a simplex Λ exists. We must show the equality. Let

$$F=\{a\in\mathbb{C}^k;\ \mathrm{Re}\, a_j>0,\ \mathrm{Im}\, a_j=0 \text{ for all } 1\leqslant j\leqslant k\}.$$

Let E be the connected component of the set $\{a\in(\mathbb{C}\setminus\{0\})^k$; there exist Λ, $z\in\Lambda\Rightarrow\langle z,a\rangle\neq 0$ and $(-1)^{k-1}(k-1)!\int_\Lambda \omega'(z)/\langle z,a\rangle^k=1/a_1\cdots a_k\}$ containing F: $F\subset E$ (by Proposition 3), $E\neq\phi$.

E is closed in $(\mathbb{C}\setminus\{0\})^k$ because for all sequences $(a_n)\to a$ with

$$(-1)^{k-1}(k-1)!\int_{\Lambda_n}\frac{\omega'(z)}{\langle z,a_n\rangle^k}=\frac{1}{a_{1,n}\cdots a_{k,n}},$$

if n is sufficiently great, we can take $\Lambda_n=\Lambda$ with Λ such that $z\in\Lambda\Rightarrow\langle z,a\rangle\neq 0$ and we can pass to the limit in the last equality.

E is open in $(\mathbb{C}\setminus\{0\})^k$ because if $\forall z\in\Lambda$, $\langle z,a\rangle\neq 0$, it is also true in a neighborhood of a. Moreover, the holomorphic functions of a,

$$(-1)^{k-1}(k-1)!\int_\Lambda\frac{\omega'(z)}{\langle z,a\rangle^k}\text{ and }\frac{1}{a_1\cdots a_k},$$

are equal in F and then are equal on the set E which is open.

Finally $E=(\mathbb{C}\setminus\{0\})^k$. ∎

PROPOSITION 5. *Let $a_1,\dots,a_k$ and Λ be as in Proposition 4. Then*

$$\frac{(-1)^{k-1}(|j|-1)!}{(j-1)!}\int_\Lambda\frac{z^{j-1}\omega'(z)}{\langle z,a\rangle^{|j|}}=\frac{1}{a_1^{j_1}\cdots a_k^{j_k}}.$$

Proof. Derivation with respect to the parameters $a_1,\dots,a_k$. ∎

THEOREM 2. *Let $a_1,\dots,a_k\in\mathbb{C}\setminus\{0\}$, $b_1,\dots,b_v\in\mathbb{C}$, Λ defined as in Proposition 4, α and j two multiindices such that $|j|>|\alpha|$. Then*

$$C_{j,\alpha}D^\alpha_{\zeta,0}\int_\Lambda\frac{z^{j-1}\omega'(z)}{[\langle\zeta,b\rangle+\langle z,a\rangle]^{|j|-|\alpha|}}=\frac{b_1^{\alpha_1}\cdots b_v^{\alpha_v}}{a_1^{j_1}\cdots a_k^{j_k}}$$

with $C_{j,\alpha}=(-1)^{|\alpha|+k-1}(|j|-|\alpha|-1)!/(j-1)!$.

Proof. The integral is well defined, because if $\langle z,a\rangle\neq 0$ then for ζ sufficiently small $\langle\zeta,b\rangle+\langle z,a\rangle\neq 0$ (ζ small uniformly with respect to $z\in\Lambda$). The equality is proved by derivation under the integral sign. ∎

Remark. The hypothesis $|j|>|\alpha|$ is not an actual restriction because it is possible to introduce an extra $a_{k+1}=1$ and an integer j_{k+1} such that $|j|+j_{k+1}>|\alpha|$.

2. Cauchy–Kovalewski Extension

For any function f defined on an open set of $\mathbb{R}^m$ we denote by f^* its Cauchy–Kovalewski extension.

Proposition 6. *Let $a_1, ..., a_k, b_1, ..., b_v$ be linear affine functions defined on $\mathbb{R}^m$ with complex values, each non-identically zero. Let j and α be two multiindices such that $|j| > |\alpha|$ and*

$$U = \{(x_0, x) \in \mathbb{R}^{m+1};\ a_l^*(x_0, x) \text{ invertible for all } 1 \leqslant l \leqslant k\}.$$

The Cauchy–Kovalewski extension of

$$\frac{b_1(x)^{\alpha_1} \cdots b_v(x)^{\alpha_v}}{a_1(x)^{j_1} \cdots a_k(x)^{j_k}}$$

is given for all $(x_0, x) \in U$ by

$$C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(x_0,x)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{|j|-|\alpha|}}.$$

$A(x_0, x)$ is a simplex of type defined in Theorem 1. It is possible to choose the simplex $A(x_0, x)$ locally independent of the point (x_0, x). The integral does not depend on the particular choice of that simplex.

Proof. $\partial a_l(x)/\partial x_i$ and $\partial b_l(x)/\partial x_i$ are constant; then their Cauchy–Kovalewski extensions are given by

$$a_l^*(x_0, x) = a_l(x) - x_0 \sum_{i=1}^{m} e_i \frac{\partial a_l(x)}{\partial x_i}$$

$$b_l^*(x_0, x) = b_l(x) - x_0 \sum_{i=1}^{m} e_i \frac{\partial b_l(x)}{\partial x_i}.$$

By Theorem 1, there exists a simplex $A(x_0, x)$ such that the integral is well defined for all $(x_0, x) \in U$, and does not depend on the choice of that simplex: for all $(y_0, y) \in U$ there exists a neighborhood $V\{(y_0, y)\} \subset U$ such that if $(x_0, x) \in V\{(y_0, y)\}$ then

$$C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(x_0,x)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{|j|-|\alpha|}}$$
$$= C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(y_0,y)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{|j|-|\alpha|}}.$$

We get a monogenic function because

$$\begin{aligned}
&\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle \\
&\quad = \sum_{l=1}^{v} \zeta_l \left(b_l(x) - x_0 \sum_{i=1}^{m} e_i \frac{\partial b_l(x)}{\partial x_i} \right) + \sum_{l=1}^{k} z_l \left(a_l(x) - x_0 \sum_{i=1}^{m} e_i \frac{\partial a_l(x)}{\partial x_i} \right) \\
&\quad = \left[\sum_{l=1}^{v} \zeta_l b_l(x) + \sum_{l=1}^{k} z_l a_l(x) \right] - x_0 \left[\sum_{i=1}^{m} e_i \left(\sum_{l=1}^{v} \zeta_l \frac{\partial b_l(x)}{\partial x_i} + \sum_{l=1}^{k} \frac{\partial a_l(x)}{\partial x_i} \right) \right] \\
&\quad = \varphi(x) - x_0 m \qquad \text{with} \quad m \in \mathscr{A}.
\end{aligned}$$

Hence,

$$\begin{aligned}
D \frac{1}{(\varphi(x) - x_0 m)^{|j| - |\alpha|}} &= (|\alpha| - |j|)\, D(\varphi(x) - x_0 m) \frac{1}{(\varphi(x) - x_0 m)^{|j| - |\alpha| + 1}} \\
&= 0.
\end{aligned}$$

For the boundary values on $\mathbb{R}^m$, we have the following. Let $(0, y) \in U \cap \mathbb{R}^m$,

$$\begin{aligned}
&\lim_{(x_0, x) \to (0, y)} C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(0,y)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{|j| - |\alpha|}} \\
&\quad = C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(0,y)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b(x)\rangle + \langle z, a(x)\rangle]^{|j| - |\alpha|}} \\
&\quad = \frac{b_1(x)^{\alpha_1} \cdots b_v(x)^{\alpha_v}}{a_1(x)^{j_1} \cdots a_k(x)^{j_k}}. \qquad \blacksquare
\end{aligned}$$

THEOREM 3. *Let $a_1, \ldots, a_k$, $b_1, \ldots, b_v$ be linear affine functions*: $\mathbb{R}^m \to \mathbb{R}$ *each non-identically zero. Let j and α be two multiindices such that $|j| > |\alpha|$ and*

$$\begin{aligned}
U = &\{(x_0, x) \in \mathbb{R}^{m+1};\, x_0 \neq 0\} \\
&\cup \{(x_0, x) \in \mathbb{R}^{m+1};\, x_0 = 0 \text{ and } a_l(x) \neq 0 \text{ for all } 1 \leqslant l \leqslant k\}.
\end{aligned}$$

The Cauchy–Kovalewski extension of

$$\frac{b_1(x)^{\alpha_1} \cdots b_v(x)^{\alpha_v}}{a_1(x)^{j_1} \cdots a_k(x)^{j_k}}$$

is given for all $(x_0, x) \in U$ by

$$C_{j,\alpha} D^{\alpha}_{\zeta,0} \int_{A(x_0,x)} \frac{z^{j-1} \omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{|j| - |\alpha|}}.$$

The harmonic extension of the pevious rational function defined on $\mathbb{R}^m$ is the real part of that Clifford-valued function.

Proof. By Proposition 6 we must study only the set of all (x_0, x) such that $a^*(x_0, x)$ is invertible. Namely, we prove that, for all (x_0, x) such that $x_0 \neq 0$, $a_l^*(x_0, x)$ is invertible:

$$\frac{\partial a_l(x)}{\partial x_i} \text{ is constant.}$$

If, for all i, this constant is 0, then $a_l(x)$ is a constant not zero (because a_l is non-identically 0) and $a_l^*(x_0, x) = a_l(x)$ is this constant. If there exists i_0 such that $\partial a_l(x)/\partial x_i$ is not 0 then

$$a_l^*(x_0, x) = a_l(x) - x_0 \sum_{i=1}^{m} e_i \frac{\partial a_l(x)}{\partial x_i}$$

is invertible for all (x_0, x) with $x_0 \neq 0$. Hence the set U. ∎

Remark. The hypothesis $|j| > |\alpha|$ is not an actual restriction.

THEOREM 4. *Let $a_1, ..., a_k$, $b_1, ..., b_v$ be linear affine functions: $\mathbb{R}^m \to \mathbb{C}$. Let $a_{k+1}, ..., a_{k+k'}$, $b_{v+1}, ..., b_{v+v'}$ be linear affine functions, $\mathbb{R}^m \to \mathbb{R}$, each of the $a_1, ..., a_{k+k'}$ non-identically zero. Let j, j', α, α' be multiindices such that $2|j| + |j'| > 2|\alpha| + |\alpha'|$ and $U = \{(x_0, x) \in \mathbb{R}^{m+1};\ (|a_l(x_0, x)|^2)^*$ invertible for all $1 \leqslant l \leqslant k$ and $a_{l'}^*(x_0, x)$ invertible for all $k+1 \leqslant l' \leqslant k+k'\}$.*

The Cauchy–Kovalewski extension of

$$\frac{|b_1(x)|^{2\alpha_1} \cdots |b_v(x)|^{2\alpha_v} b_{v+1}^{\alpha'_1}(b)_{v+v'}^{\alpha'_{v'}}(x)}{|a_1(x)|^{2j_1} \cdots |a_k(x)|^{2j_k} a_{k+1}^{j'_1}(x) \cdots a_{k+k'}^{j'_{k'}}(x)}$$

is given for all $(x_0, x) \in U$ by

$$C_{2j+j', 2\alpha+\alpha'} D_{\zeta,0}^{\alpha} \int_{A(x_0, x)} \frac{z^{j-1}\omega'(z)}{[\langle \zeta, b^*(x_0, x)\rangle + \langle z, a^*(x_0, x)\rangle]^{2|j| + |j'| - 2|\alpha| - |\alpha'|}}$$

with

$$\langle \zeta, b^*(x_0, x)\rangle = \sum_{l=1}^{v} \zeta_l b_l^*(x_0, x) + \zeta_{v+l} \bar{b}_l^*(x_0, x) + \sum_{l=1}^{v'} \zeta_{2n+l} b_{v+l}^*(x_0, x)$$

$$\langle z, a^*(x_0, x)\rangle = \sum_{l=1}^{k} z_l a_l^*(x_0, x) + z_{k+l} \bar{a}_l^*(x_0, x) + \sum_{l=1}^{k'} z_{2v+l} a_{k+l}^*(x_0, x).$$

The harmonic extension of the previous rational function defined on $\mathbb{R}^m$ is the real part of that Clifford-valued function.

Proof. By Proposition 6 we must show only $\{(x_0, x) \in \mathbb{R}^{m+1};\ a_l^*(x_0, x)$ and $\bar{a}_l^*(x_0, x)$ invertible for all $1 \leqslant l \leqslant k\} = \{(x_0, x) \in \mathbb{R}^{m+1};\ (|a_l(x_0, x)|^2)^*$ invertible for all $1 \leqslant l \leqslant k\}$.

Hence, we must show only

$$a_l^*(x_0, x) \text{ and } \bar{a}_l^*(x_0, x) \text{ invertible} \Leftrightarrow (|a_l(x_0, x)|^2)^* \text{ invertible.}$$

An element σ of a complexified Clifford algebra and its complex conjugate $\bar{\sigma}$ are such that σ and $\bar{\sigma}$ invertible $\Leftrightarrow$ $\sigma\bar{\sigma}$ invertible; then

$$a_l^*(x_0, x) \text{ and } \bar{a}_l^*(x_0, x) \text{ invertible} \Leftrightarrow a_l^*(x_0, x)\, \bar{a}_l^*(x_0, x) \text{ invertible.}$$

But we must show that the extension of $|a_l(x)|^2$ is precisely that product:

$$(|a_l(x_0, x)|^2)^* = a_l^*(x_0, x)\, \bar{a}_l^*(x_0, x)$$

$$a_l^*(x_0, x)\, \bar{a}_l^*(x_0, x) = a_l \bar{a}_l - a_l x_0 \sum e_i \frac{\partial \bar{a}_l}{\partial x_i} - \bar{a}_l x_0 \sum e_i \frac{\partial a_l}{\partial x_i}$$

$$+ x_0^2 \sum e_i e_{i'} \frac{\partial a_l}{\partial x_i} \frac{\partial \bar{a}_l}{\partial x_{i'}}.$$

This is monogenic function on $\mathbb{R}^{m+1}$ which is equal to $|a_l(x)|^2$ on $\mathbb{R}^{m+1}$. The Cauchy–Kovalewski extension is unique; then we get the inequality. ∎

Remark. Intuitively the singularities do not see the non-commutativity structure.

3. A Connection between Complex Analysis and Clifford Analysis

In [4], some connections between the two analyses were shown. Here we use the previous ideas to construct monogenic functions starting from holomorphic functions in a local manner.

Define the kernel

$$K(z, x_0, x) = (-1)^{k-1}(k-1)! \int_{A(z, x_0, x)} \frac{\omega'(\zeta)}{\langle \zeta, z - (x - ex_0) \rangle^m}$$

with

$$\langle \zeta, z - (x - ex_0) \rangle = \sum_{j=1}^{m} \zeta_j [z_j - (x_j - e_j x_0)].$$

$A(z, x_0, x)$ is a simplex of the type defined in Theorem 1. This theorem shows that this kernel is defined for all z such that $z_j-(x_j-e_j x_0)$ is invertible; hence it is defined for all z such that $z_j \neq x_j \pm i x_0$.

Let $W=\{z \in \mathbb{C}^m; \operatorname{Im} z_1 = \pm\operatorname{Im} z_2 = \cdots = \pm\operatorname{Im} z_m\}$. W is a 2^{m-1} covering of $\mathbb{R}^{m+1}$,

$$(x_1 \pm i x_0, ..., x_m \pm i x_0) \mapsto (x_0, x_1, ..., x_m).$$

Then, we have the following.

PROPOSITION 7. *Let f be a holomorphic function defined in a neighborhood of W. Let*

$$P=\{z \in \mathbb{C}^m; |z-(x_j \pm i x_0)| < r_j \text{ for all } 1 \leqslant j \leqslant m\}$$

with r_j sufficiently small; then

$$F(x_0, x)=\int_{\partial P} f(z)\, K(z, x_0, x)\, d\sigma(z),$$

where ∂P is the distinguished boundary of P and $d\sigma$, the standard measure on it, is a monogenic function.

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